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DIFFERENCES BETWEEN CBAR & CBEAM ELEMENTS

1. INTRODUCTION

The scope of this document is to clarify the differences between the CBAR and the CBEAM element.

In the first chapter, some generic topics of both elements are going to be discussed such as offsets or stress recovery coefficients.
In the second chapter, the way in which each of the force component affect the CBAR and the CBEAM elements is going to be explained.
Before comparing both elements, the position and the influence of the pin flags will be explained.
Afterwards, taking the results from the third chapter, the comparison of both models will be performed.
Once all the previous topics are explained, the possible errors for offset CBEAM or CBAR elements are discussed. A typical example of this situation is the stringer-skin configuration in an aircraft.
Finally, in order to have a deeper understanding of NASTRAN as well as to clarify better the previous concepts, the stiffness and flexibility matrixes are explained.

It is true that the use of CBAR elements has decreased due to the increment of computing efficiency. But despite this fact, its use is going to be explained so as to discern when it is correct to use CBAR and also to obtain a better understanding of NASTRAN.

2. THEORETICAL BEHAVIOUR OF CBEAM & CBAR

Firstly, in order to have the necessary knowledge for the next chapters, some definitions are going to be introduced:

- Shear centre (sc): it is an imaginary point on the section, where a shear force can be applied without inducing any additional torsion.
- Centroid: Also known as the geometric centre, it is the arithmetic mean position of all the points in the shape (assuming a material with homogeneous modulus of elasticity). It can be obtained as the division between the first moment of area and the total area of the section. When an axial force is applied in this point, no bending moments will appear (similar to shear centre with shear force and torque). Usually it is called gravity centre because most of the times they are coincident.
- Gravity centre (gc): it is the point at which the entire weight of the section may be considered as concentrated so that if supported at this point the section would remain in equilibrium.

Under some specific conditions, the previous points may be coincident. The shear centre and the centroid are coincident when the section has two axes of symmetry.
The centroid and the gravity centre are coincident when the mass distribution of the section is homogeneous. It is possible to have non-coincidence when non-structural mass is used.

- CBAR

The CBAR element represents a one dimensional element which simulates the classical beam stiffness. That is, axial, bending, shear and torsion stiffness. But there are some limitations which are shown below:

- The beam properties must be constant along the length of the element.
- The shear centre, the centroid, and the gravity centre must be coincident.
- It is not possible to take into account the influence of warping on the torsional rotation.

One of the important aspects of the CBAR are the stress recovery coefficients. These coefficients define the y and z components of the points in which stresses are computed. The y and z values are given in the element coordinate system. The coefficients must be input in the fields from 2 to 9 of the second line of the PBAR card. An example of a PBAR is plotted below(Points: 1 (49.5, 0.5), 2 (-0.5, 99.5), 3 (-0.5, -0.5), 4 (49.5, -0.5)):

```
PBAR    1       1       149.    40437.4 328362. 49.6667
        49.5    .5      -.5     99.5    -.5     -.5     49.5    -.5
        .328859 .66443  .1
```

As it will be discussed later, some care must be taken when the stresses recovery coefficients are used.

Another relevant aspect of the CBAR element is the offset. The offset value must be introduced in the 4 to 9 fields of the second line of the CBAR card. By default, the offset values are measured in the analysis coordinate system of each grid (the so called, global system of NASTRAN). An example of a CBAR card is shown below (Offset = (0., 8.2214, 33.221)):

```
CBAR    1       1       1       2       0.      1.      0.
                0.      8.2214  33.221  0.      8.2214  33.221
```

- CBEAM

DIFFERENCES BETWEEN CBAR & CBEAM ELEMENTS

The CBEAM element is an improvement of the CBAR element. That is, the deficiencies of the CBAR are fixed.

- It is possible to have variable properties along the length. Although this option is normally replaced by a finer mesh with constant properties per element.
- The shear centre, the centroid and the gravity centre can be moved independently.
- The influence of warping on the torsional rotation can be taken into account. The effect of the warping on the section rotation due to torque is a second order effect normally neglected except for open sections.
- The shear relief effect is considered. This effect is only accounted for variable sections and therefore not typically taken into account.

The offset and the stress recovery coefficients are used and introduced for the CBEAM element in the same way as they are for the CBAR element.

Finally, before moving onto next chapter the NASTRAN sign criterion is going to be displayed so as to clarify them for future uses.

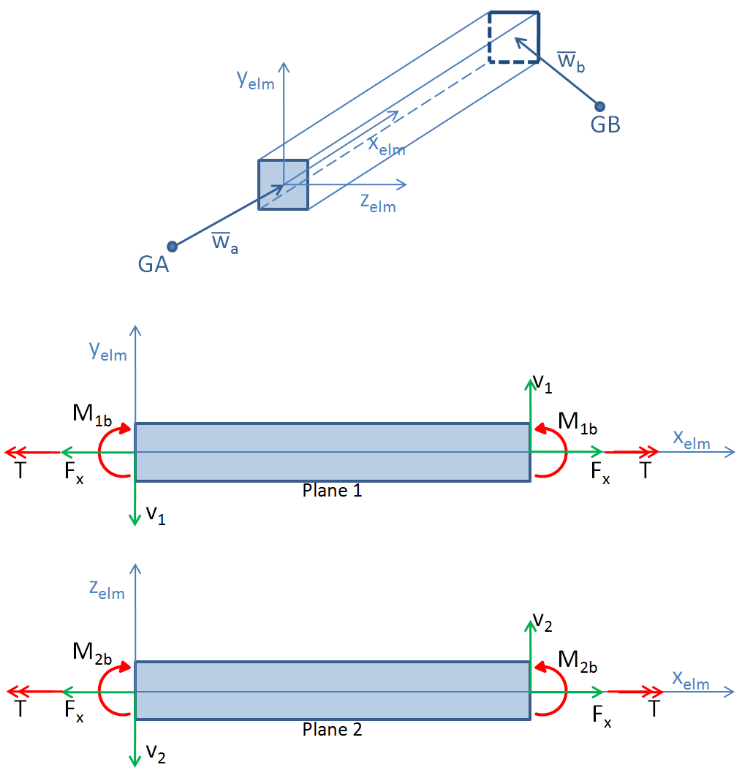


Figure 1: NASTRAN 1D element sign criterion.

It is important to highlight that for both elements (CBAR and CBEAM), the offset vectors

(w_a and w_b) move the shear centre with regards to the nodes.
For the CBEAM element, it is possible to move the centroid and the gravity centre independently. To achieve it, the 2 to 9 fields of the sixth line of the PBEAM card must be filled. An example of this is shown below (gravity centre offset (1.5546, 5.2648), centroid offset (8.22148, 33.2215)):

```
PBEAM    1       1       149.    30366.1 163916. -40696.349.6667
         49.5    .5      -.5     99.5    -.5     -.5     49.5    -.5
         YES     1.      149.    30366.1 163916. -40696.349.6667
         49.5    .5      -.5     99.5    -.5     -.5     49.5    -.5
         .328859 .66443                          30732.3 30732.3
         1.5546  5.2648  1.5546  5.2648  8.22148 33.2215 8.22148 33.2215
```

Further information can be found in the Quick Reference Guide of NASTRAN.

3. LOADS INFLUENCE

The influence of each load component on the beam is going to be explained in the following paragraphs. The results for each load in both elements (CBAR and CBEAM) are compared.

In order to obtain the most general results, a section with non-coincident shear centre and centroid is going to be analysed.

On NASTRAN, the loads are introduced in the elements through the nodes. And when there is an offset, the internal loads on the elements coming from the external loads on the nodes are computed translating the forces/moments from the node to the element axis applying equilibrium of forces and moments. The computation of the loads considers the offset as a rigid element (a kind of internal RBE2).

The section which is going to be studied is displayed below:

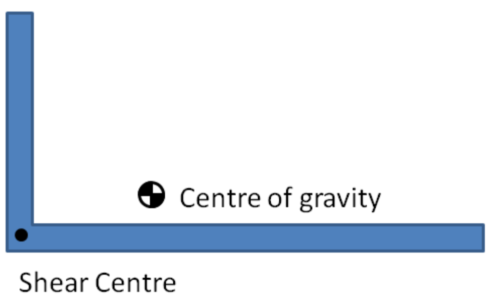


Figure 2: CBEAM section to be analysed.

- F_x (axial force)

CBEAM

The axial force applied on a CBEAM element is transferred through the nodes. If there is no offset, the nodes and the shear centre will be coincident. Consequently, there is an offset between the node and the centroid of the section. As a result of it, the internal loads on the beam element will consist of an axial force and bending moments. These internal moments are generated due to the distance between the node and the centroid.

As a result of these loads, flat distributions of stresses and strains on the cross sectional area will be obtained ($\sigma = a \cdot y + b \cdot z + c$). As it was discussed before, these results can be computed in the stress recovery points (C, D, E, and F). For the CBEAM element, the stress recovery points must be referenced to the shear centre (origin of the axis system) and must be expressed on the element coordinate system.

An example of the stresses reported in .f06 file for a CBEAM element is displayed below. It is important to notice that the stresses values represent the sum of the axial and the bending contributions.

```
                          S T R E S S E S   I N   B E A M   E L E M E N T S        ( C B E A M )
                    STAT DIST/
ELEMENT-ID  GRID    LENGTH    SXC           SXD           SXE           SXF           S-MAX         S-MIN         M.S.-T   M.S.-C
      1
               1    0.000   -1.360555E+00 -1.300774E+00  2.744000E+00 -1.320107E+00  2.744000E+00 -1.360555E+00
               2    1.000   -1.360555E+00 -1.300774E+00  2.744000E+00 -1.320107E+00  2.744000E+00 -1.360555E+00
```

If the section analysed has two symmetric planes, then the centroid and the shear centre will be coincident, therefore the axial force will not create bending moments if no offset is included.

When an offset is used, the axial force will create bending moments in the centroid.

CBAR

The use of a CBAR element in order to represent this kind of section has some drawbacks. For CBAR elements the shear centre and the centroid are coincident. Therefore, the axial-bending coupling relation is the same that the shear-torsion coupling relation. This gives as a consequence that the loads applied in the nodes are applied in the centroid (if an offset is not provided).

In order to simulate a coupling, the distance between the node (load application point) and the centroid=shear center should be introduced by means of an offset. This offset must translate the section so as to position the centroid in the correct place with regards to the node (load application point).

In the following figure it is possible to see a sketch of the previous concept. It is important to notice that the CBAR section has not been represented as the real section, as the shear center and the centroid are coincident on the CBAR and not on the real section.

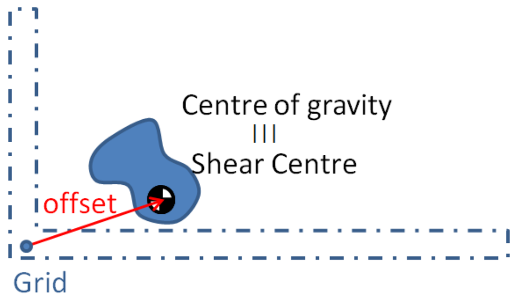


Figure 3: Sketch of CBAR offset translation.

The result of this modification is that the bending moments obtained in the centroid are the same as the ones obtained in the CBEAM element. Therefore the displacements, stresses and strains are the same for both elements as only axial loads are included.

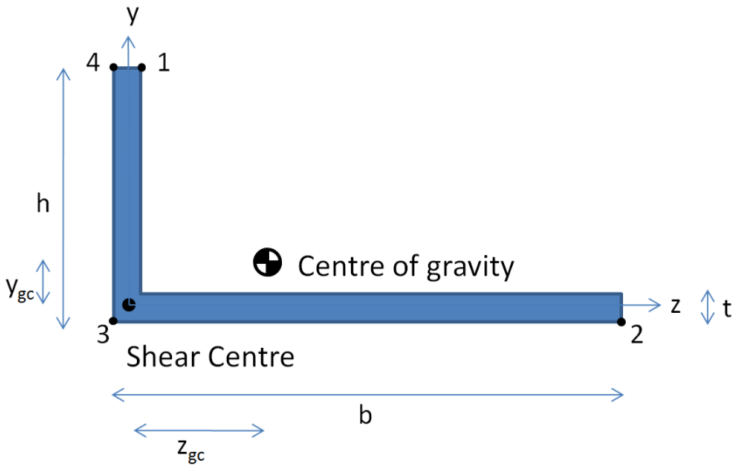
Regarding the stress recovery coefficients, it is important to highlight that they must be referenced to the centroid of the real section in order to obtain the same stresses and strains as the ones obtained in the CBEAM element as the stresses are axial stresses.

An example of the CBAR stresses is plotted below. It is important to highlight that the axial and bending components are displayed separately.

S T R E S S E S   I N   B A R   E L E M E N T S   (C B A R)

ELEMENT ID.	SA1 SB1	SA2 SB2	SA3 SB3	SA4 SB4	AXIAL STRESS	SA-MAX SB-MAX	SA-MIN SB-MIN	M.S.-T M.S.-C
1	-2.031694E+00	-1.971915E+00	2.072858E+00	-1.991247E+00	6.711410E-01	2.743999E+00	-1.360554E+00	
	-2.031694E+00	-1.971915E+00	2.072858E+00	-1.991247E+00		2.743999E+00	-1.360554E+00	

Below are plotted the coordinate of the stress recovery points, referenced to the shear center and to the centroid.



Points\Reference	Shear center (y)	Shear center (z)	Centroid (y)	Centroid (z)
1	h-t/2	t/2	h-t/2-y_{cg}	t/2-z_{cg}
2	-t/2	b-t/2	-t/2-y_{cg}	b-t/2-z_{cg}
3	-t/2	-t/2	-t/2-y_{cg}	-t/2-z_{cg}
4	h-t/2	-t/2	h-t/2-y_{cg}	-t/2-z_{cg}

- My & Mz (Bending moments)

BEAM

In this case, the bending moments, which are sliding vector, give as a result a flat stress distribution. For this configuration, the neutral axis (zero strain axis) is coincident with the centroid because there is no axial force. There is no influence of the offset on the internal forces of the element.

BAR

The stresses and strains obtained in this element are exactly the same as the ones obtained in the CBEAM element. This is due to the bending moments are sliding vectors.
As a consequence, the element forces are also equal for both elements.
On the other hand, the displacements obtained are not the same. The translations and rotations in the plane 1 and 2 are equal for both elements, but the axial displacement is different. The reason of this difference is that the rotations in plane 1 and 2 give as a result axial displacements on the different points of the cross section. This displacement depends on the distance between the centroid and the nodes (on NASTRAN the displacements are computed at the nodes). If the node and the centroid are not coincident, the value of the displacement on the node will be the value of the displacement at the point of the cross section where the node is located.

DIFFERENCES BETWEEN CBAR & CBEAM ELEMENTS

For the CBEAM element the nodes and the shear center are coincident, but the centroid position is different. On the other hand, for the CBAR element, the nodes, the shear center, and the centroid are coincident. Therefore, the strain in the axial direction will be zero on the CBAR nodes. Consequently, the displacement on the CBAR is the axial displacement on the centroid (null), but the displacement on the CBEAM is the axial displacement on the shear center (which is not null). Thus, in order to obtain the desired displacement, it is necessary to have an eccentricity between the nodes and the centroid. To obtain the same results as the ones obtained in the CBEAM element, it is necessary to include an offset which represents the distance from the shear center to the centroid of the real section.
A sketch of the previous concept is shown below:

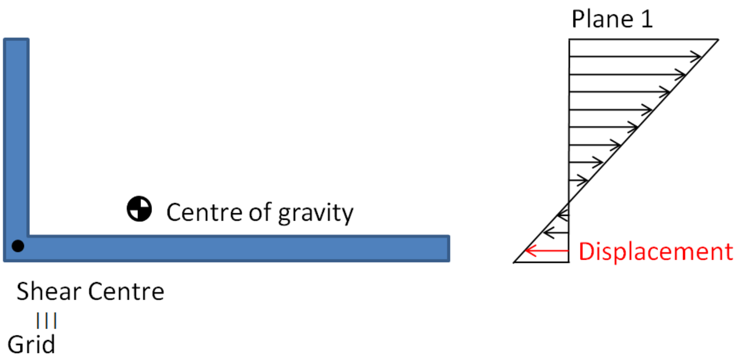


Figure 4: CBEAM displacement.

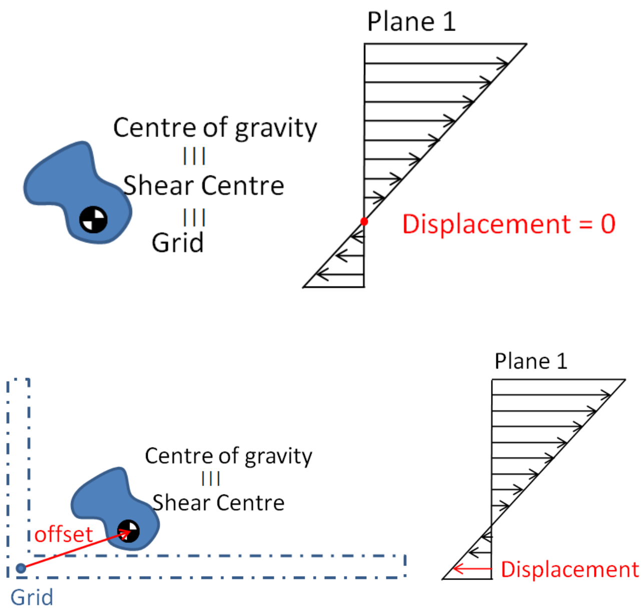


Figure 5: CBAR displacement.

- Fy & Fz (Shear forces)

<u>BEAM</u>
The external loads are applied on the nodes. In the case of the CBEAM element, if there is no offset, the nodes and the shear center are coincident. In this case the shear forces do not create torque.
When an offset is introduced, a torque can be produced due to the in section-plane offset and also it is possible to obtain bending moments due to out of section-plane offset.

For this kind of forces, shear stiffness coefficients can be introduced by mean of field 2 and 3 of the fifth line of the PBEAM card. These coefficients are only taken into account when the inertia coefficient I_{12} is null. The shear stiffness coefficients create some shear deformation on the beam due to transversal loads in a way similar to the axial loads create axial deformation (for the Euler-Bernoulli beams theory this shear deformation is neglected). The values are just a kind of efficiency of the cross sectional area with shear loads. Contrary to the axial loading (the cross sectional area works with constant stress), the shear loading creates a specific distribution of shear stresses on the cross sectional area (parabolic for square sections; shown in blue on the figure below). Consequently, the stiffness provided by the area is not 100%, but a lower value (shown in red on figure below). This value is dependent on the shape of the cross section and it is normally tabulated according to the cross-section shape. For square sections the value is 0.8333.

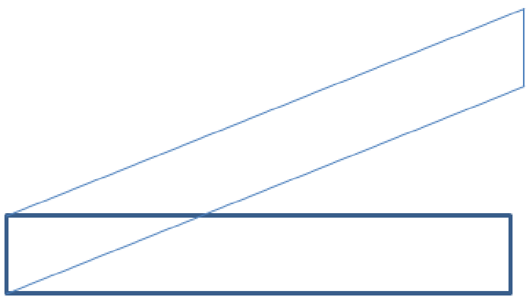


Figure 6: Shear deformation of a CBEAM

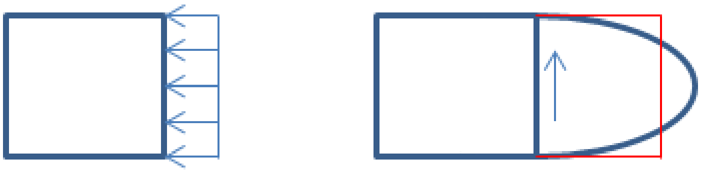


Figure 7: Axial and Shear Stress distribution on a square cross section of a CBEAM

These coefficients can also be defined for CBAR element.

BAR

As it happens for the previous element, the shear forces are applied in the shear center Therefore all the results (stresses, strains, displacement, element forces), with the stress recovery coefficient referenced to the centroid, are almost the same. There is only one exception for the results, this is the axial displacement. This displacement is the result of the plane 1 and 2 rotations.

To fix this discrepancy, it is possible to introduce an offset so as to have the correct distance between the centroid and the nodes. This problem is the same as the one of the bending moment topic.

But trying to fix the previous problems leads as a result another problem because introducing the previous offset will create a torque which will change the transversal displacements.

- Mx (Torque)

BEAM

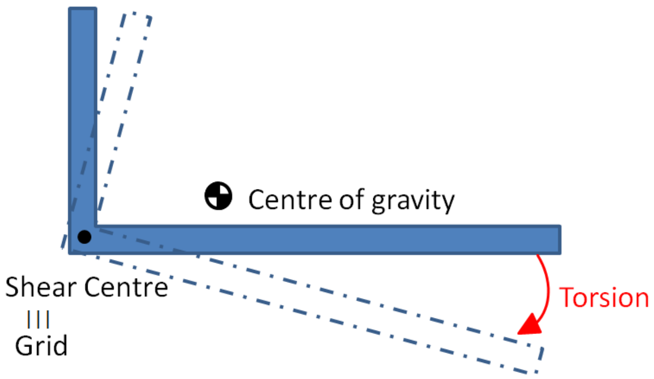
As the torque is a sliding vector, it does not create any other component. This structure will be solved as the general theory will do.

Note that NASTRAN only computes axial stresses. Therefore for this example the stresses will be null.

If there is no offset, the torque will only generate torsion since the shear center and the nodes are placed in the same position.

On the other hand, if there is offset, the torque will not only generate torsion but also transversal displacements on the nodes. The reason is that the transversal displacements on points on the cross section different to the shear center are not null (similar to the axial displacements on the cross section on points different to the centroid).

The previous concept is explained in the following figure.



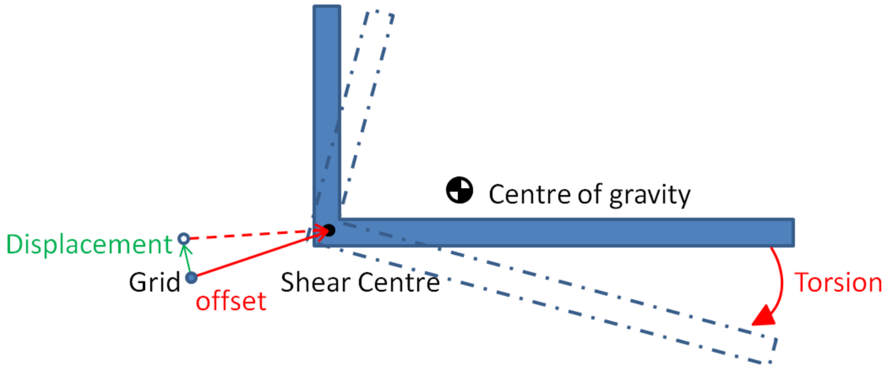


Figure 8: Offset versus no-offset torsion

In the no-offset case, the rotation around the shear center does not generate any displacement. But for the offset case, the rotation translates the node.

BAR
For the CBAR element, the results obtained are exactly the same than the ones obtained for the CBEAM element.

4. PIN FLAGS BEHAVIOUR

Pin flags can be used in the CBAR and CBEAM cards so as to release the selected DOF. The DOF to be disconnected are introduced in the fields 2 and 3 of the second line. In the following NASTRAN code, the DOF 4, 5 and 6 are disconnected in the nodes 1 and 2 of the CBAR.

```
CBAR    1     1     1     2       0.    1.      0.
        456   456   0.    8.2214  33.221  0.    8.2214  33.221
```

The pin flags not only remove the stiffness components from the elements of the selected DOF, but also modify other components of the stiffness matrix.
The components modified depend on the coupling between the different DOF. For example, for the simplest structure (no offset and shear center and centroid coincident) the release of the axial DOF leads to a modification of the axial DOF of both nodes. But, on the other hand, when there is coupling between the axial and the bending components, these components will be modified by the use of pin flag in the axial DOF.
This can be generalised for the remaining components.

But, the main purpose of this chapter is to clarify the position of the pin flags when offsets are used.

For example, let's say that the rotation (perpendicular to the sheet) DOF of the structure shown below is release by means of a pin flag in the right node.

Figure 9: CBEAM/CBAR structure with pin flags.

If the structure is loaded by means of a rotation in the pin flagged node, the behaviour will be different if the pin flag is placed in the bottom or on the tip of the offset.
If the pin flag were in the bottom of the offset (node), the rotation will not load the structure.
On the other hand, if the pin flag were in the tip of the offset (shear center), the structure would be loaded with an axial displacement. The two possible configurations are shown in the following sketch.

Figure 10: Pin flag in the bottom

Figure 11: Pin flag in the top

The difference between the locations of the pin flag has been explained in order to understand its relevance. But there is only one real configuration. In the NASTRAN code, the pin flags are placed in the tip of the offset (shear center) and not basis of the offset (node), so most of the coupling effects (due to offset) are still taken into account despite the use of pin flags.

To summarise the idea and not be confused by the topics explained before, the following sketch represent the NASTRAN position of pin flags.

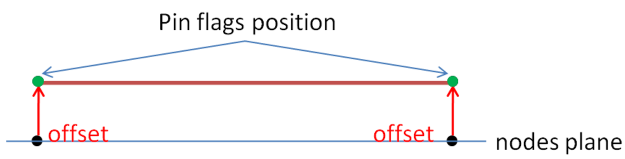


Figure 12: Pin flag position

5. CBEAM versus CBAR

This chapter deals with the differences between both elements when all kind of loads (axial, bending, shear and torque) are applied on them.

As the shear center and the centroid for the CBAR element are coincident, the element can be offset in order to place one of the two points in the desired place but not both at the same time in general. On the CBEAM, both points can be correctly located as there is an additional offset (from shear center to centroid) that can be used to properly locate the centroid of the section.

If the section analysed has two planes of symmetry, then both elements could be used for the same purpose. But, it is important to notice that the CBEAM element can take into account different aspects such as shear relief or section variation.

For a generic section, the following two possibilities apply.

- **Properly positioned shear centre on the CBAR, incorrect position of the centroid on the CBAR versus CBEAM with the shear centre and centroid properly positioned**

In this case, the shear centre on the CBAR element is coincident with the real position of the shear centre on the real structure.
By default, when the offset is null, the shear center in the real section are coincident with the nodes. This is the situation for both, the CBEAM and the CBAR element by default.
With this configuration, the shear forces as well as the torque are correctly computed.

Comparing the results for the CBAR and the ones for the CBEAM, the following conclusions will be obtained:

- Displacements: Only the torsional rotation is the same on both elements. The reason is that the shear center is in the same position on both elements (no different shear-torque coupling on CBEAM or CBAR). The axial displacements can be different due to the centroid location. The transverse displacements can differ due to the bending coming from the coupling with the axial forces.
- Element Forces: The shear forces and the torque are the same on both elements. The reason is that the shear center is in the same position on both elements (no different shear-torque coupling on CBEAM or CBAR). The axial forces can be different due to the centroid location and the transverse moments as well due to the bending coming from the coupling with the axial forces.
- Stresses and Strains: Only the axial contribution (axial stress/strain due to axial force)

is the same for both elements. There is an additional component coming from the bending that is different because of the different axial-bending coupling.

- **Properly positioned centroid on the CBAR, incorrect position of the shear center on the CBAR versus CBEAM with the shear centre and centroid properly positioned**

In this case the centroid on the CBAR element is placed in its correct place by using an offset. Doing this, the shear center is also translated placing it at an incorrect position.

Comparing the results from the CBAR to the ones from the CBEAM, the following topics will be obtained:

- Displacements: Longitudinal translation and rotations in plane 1 and 2 are the same for both elements. These magnitudes are related with the axial-bending coupling that is correctly represented on the CBAR.
- Element Forces: All the forces are the same except the torque. The shear-torque coupling is incorrectly represented on this CBAR as the shear center is incorrectly positioned.
- Stresses and Strains: Identical for both elements. These values are related with the longitudinal direction (axial and bending).

Therefore, the main conclusions are:

- Whenever possible using CBEAM leads to higher accuracy as the shear center and centroid are correctly positioned.
- If CBAR is used the engineer has to choose what effect is more important on the model, the shear-torque coupling or the axial-bending coupling. He has to choose what point has to be correctly positioned (shear centre or centroid) to minimise the error on the important magnitudes. In general, the axial-bending coupling uses to be the most important relation. Therefore, normally the centroid will be correctly positioned neglecting the errors due to the incorrect location of the shear center.

6. POSSIBLE OFFSET PROBLEMS

In this chapter, some of the errors when an offset is used are going to be explained.
The ways to fix these problems are also listed.

One of the most typical situations in which this problem exists, is the stringers and skin model.

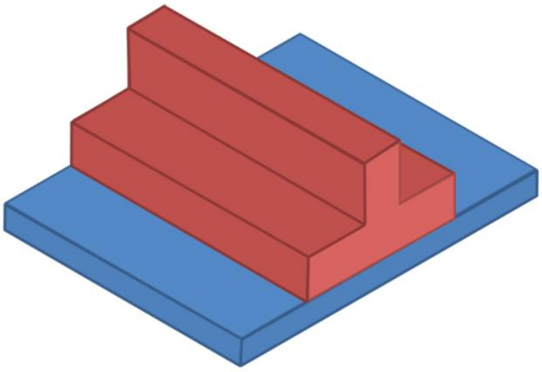


Figure 13: Skin and stringer configuration

Usually, the nodes of the model are placed in the mean plane of the skin or even in the outer plane. Therefore in order to model the real configuration, the stringer must be offset or modified.

The most common and intuitive situation is the use of offset. As it can be seen in the following figure, the offset translate the CBEAM or the CBAR element to its correct place.

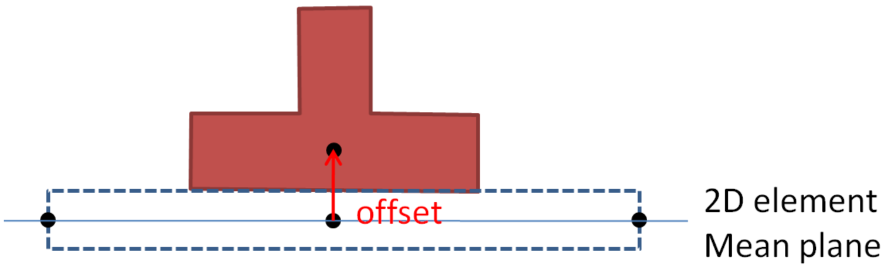


Figure 14: CBEAM/CBAR offset sketch

In this situation, the loads that flow through the mean plane nodes will be translated to the offset position adding the appropriate moments as it has been explained in the previous chapters (by an static equivalency).

But let's think about the stiffness of the model and how it should change in order to represent the stiffness of the real structure.

Analysing this structure by hand, the inertia of the stringer must be referenced to the node of the skin mean plane. To compute easily the moments, the Steiner theorem should be used. This theorem compute the moment referenced in one axis based on the moment calculated at the centroid. It can be seen in the following figure.

$$I_{x'} = I_x + A \cdot l^2$$

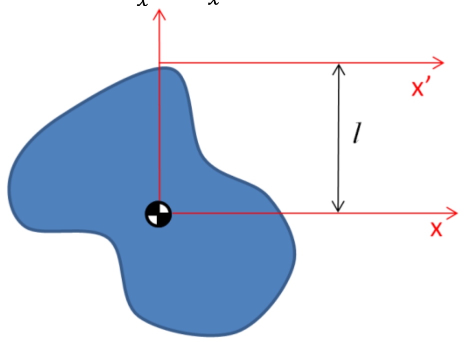


Figure 15: Steiner sketch

This modification of the moments of inertia will lead as a consequence to a higher stiffness for the shear-bending and bending coefficients. But, it will not include the effect of the eccentricity of the centroid of the skin+stringer with regards to the node (load application point).

Therefore, it is possible to model the real structure without using an offset, using the Steiner formula with which the PBAR/PBEAM properties will be modified. But, this way to model the real structure will have important inaccuracies with regards to axial-bending coupling, and therefore, with regards to the global stiffness represented on the model. However, the flexibility of the Bending moment –> bending rotation of the skin+stringer is correctly matched. On the other hand, the stiffness bending rotation->Bending moment is not correctly matched as will be demonstrated hereafter.

Now let's move onto the stiffness of the structure with an offset. An offset is equal to a rigid element, so the best way to understand how it works is to visualize the rigid element.

The side view of the offset example is displayed below:

Figure 16: Beam offset sketch

The stiffness of the structure can be related to the loads and displacements by means of the following equation:

$$\{P\} = [k]\{\delta\}$$

Therefore, the loads obtained when a unitary displacement (with zero displacement on all the other DOF) is introduced will represent some of the stiffness matrix's components.

In this case, the desired stiffness component is the one which relates the bending with the rotation.

To determine the previous relation, a rotation is introduced in the structure studied. As the offset behaviour is like a rigid element, the following sketch represents the displacements.

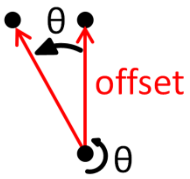


Figure 17: offset rotation sketch

As it can be seen, a rotation in the lower node generates a rotation and a displacement in the upper one.

Assuming small displacements it is possible to say:

$$\sin(\theta) = \tan(\theta) = \theta$$
$$\cos(\theta) = 1$$

Therefore, the displacement of the upper node is the multiplication of θ and the offset. That is the longitudinal displacement of the beam. Both, the displacement and rotation applied to the beam are displayed below:

Figure 18: Displacements in the beam

As a reaction to these displacements, with the beam clamped on its other side, shear, bending and axial loads will be obtained.

The loads obtained are shown below:

$$Q_{shear} = \frac{6EI}{L^2}\theta$$
$$M_{bending} = \frac{4EI}{L}\theta$$
$$F_{axial} = \frac{EA}{L}\delta = -\frac{EA}{L}\cdot\theta\cdot offset$$

The internal forces above are the ones applied on the tip of the offset, but to obtain the stiffness of the matrix, they must be computed on the basis node of the offset.

Therefore, translating them from the upper to the lower node, the shear and the axial forces will be the same. But the bending moment will take into account the moment generated by the axial force. That is explained below:

$$M'_{bending} = M_{bending} - F_{axial} \cdot offset$$

$$M'_{bending} = \frac{4EI}{L}\theta + \frac{EA}{L} \cdot \theta \cdot offset^2$$

$$M'_{bending} = \theta \frac{4E}{L}\left[I + A \cdot \left(\frac{offset}{2}\right)^2\right]$$

So, the bending stiffness is the relation between the moment and the rotation in the original node:

$$\frac{M'_{bending}}{\theta} = \frac{4E}{L}\left[I + A \cdot \left(\frac{offset}{2}\right)^2\right]$$

The resultant moment of inertia is equal to the moment of inertia of the beam plus a component that can be interpreted as a Steiner modification with half of the offset.

Therefore, the stiffness Bending Moment-> bending rotation obtained is higher than the one without offset, but lower than the one obtained with Steiner with the whole offset distance.

The model which uses Steiner does not take into account axial-bending or shear-torsion relations which are produced by the offset. For example, if the section analysed has an axial load applied, the stress distribution will be uniform.

On the other hand, for the model with offset, the skin stress distribution through the thickness will be linear while the stress distribution for the stringer will be linear as well as there is a moment (axial load · offset).

Finally, to obtain the stresses and strains properly, for the Steiner model, the stress recovery coefficients must be referenced to the mean plane node position. Their value will be the same as the model with offset coefficients plus the offset vector.

7. STIFFNESS AND FLEXIBILITY MATRIXES

First of all, it is important to clarify the differences between both matrixes.
Both matrixes relate the forces and the displacements in the structure, but one is the inverse of the other.
The stiffness matrix is represented by means of a $[K]$ and the flexibility by means of a $[C]$.
The relation of both matrixes are displayed below:

$$\{P\} = [K]\{\delta\}$$
$$\{\delta\} = [C]\{P\}$$

Where $\{P\}$ is the load vector and $\{\delta\}$ is the displacement vector.
The way to obtain the stiffness matrix can be found in the AERSYS knowledge unit 7003.
Although it is not the most useful way, loading the structure with unitary displacements is the most straightforward way to obtain it.
The best way to understand it, is taking a look on the stiffness matrix of a beam. For a prismatic beam with Ixy null, the matrix is as follows.

$$
\begin{array}{c}
\begin{array}{cccccccccccc}
u_1 & v_1 & w_1 & \theta_{x1} & \theta_{y1} & \theta_{z1} & u_2 & v_2 & w_2 & \theta_{x2} & \theta_{y2} & \theta_{z2} \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow
\end{array} \\
[K]= \left|\begin{array}{cccccccccccc}
\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\
 & & \frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 & 0 & 0 & -\frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 \\
 & & & \frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 \\
 & & & & \frac{4EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\
 & & & & & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \\
 & & & & & & \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & \frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} \\
 & & & \text{Sym.} & & & & & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\
 & & & & & & & & & \frac{GJ}{L} & 0 & 0 \\
 & & & & & & & & & & \frac{4EI_y}{L} & 0 \\
 & & & & & & & & & & & \frac{4EI_z}{L}
\end{array}\right|
\end{array}
$$

If it is desired to take into account the shear stiffness factor, the following changes must be done in the previous matrix (for both, I_y and I_z):

$$\frac{12EI_y}{L^3} \to R$$

$$\frac{6EI_y}{L^2} \to \frac{L}{2}R$$

$$\frac{4EI_y}{L} \to \frac{L^2}{4}R + \frac{EI_y}{L}$$

$$\frac{2EI_y}{L} \to \frac{L^2}{4}R - \frac{EI_y}{L}$$

Where:

$$R = \left(\frac{L}{K_z AG} + \frac{L^3}{12EI_y}\right)^{-1}$$

But to make it clearly, let's use a numerical example:

```
PBEAM   1       1       100.    833.333 833.333 0.      1408.33
        5.      5.      -5.     5.      -5.     -5.     5.      -5.
        YES     1.      100.    833.333 833.333 0.      1408.33
        5.      5.      -5.     5.      -5.     -5.     5.      -5.
        .833333 .833333                         0.      0.
                                0.      0.      0.      0.
CBEAM   1       1       1       2       0.      1.      0.
                        0.      0.      5.5     0.      0.      5.5
MAT1    1       70000.          .3
GRID    1               0.      0.      0.
GRID    2               100.    0.      0.
```

To simplify the process, a square section is used, with the properties shown in the previous NASTRAN code.

With those values, the stiffness matrix without offset can be computed, giving as a result:

$$[K]=\begin{array}{c}
\begin{array}{cccccccccccc}
u_1 & v_1 & w_1 & \theta_{x1} & \theta_{y1} & \theta_{z1} & u_2 & v_2 & w_2 & \theta_{x2} & \theta_{y2} & \theta_{z2} \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow
\end{array} \\
\left|\begin{array}{cccccccccccc}
7.00E+4 & 0 & 0 & 0 & 0 & 0 & -7.00E+4 & 0 & 0 & 0 & 0 & 0 \\
0 & 6.79E+2 & 0 & 0 & 0 & 3.39E+4 & 0 & -6.79E+2 & 0 & 0 & 0 & 3.39E+4 \\
0 & 0 & 6.79E+2 & 0 & -3.39E+4 & 0 & 0 & 0 & -6.79E+2 & 0 & -3.39E+4 & 0 \\
0 & 0 & 0 & 3.79E+5 & 0 & 0 & 0 & 0 & 0 & -3.79E+5 & 0 & 0 \\
0 & 0 & -3.39E+4 & 0 & 2.28E+6 & 0 & 0 & 0 & 3.39E+4 & 0 & 1.11E+6 & 0 \\
0 & 3.39E+4 & 0 & 0 & 0 & 2.28E+6 & 0 & -3.39E+4 & 0 & 0 & 0 & 1.11E+6 \\
-7.00E+4 & 0 & 0 & 0 & 0 & 0 & 7.00E+4 & 0 & 0 & 0 & 0 & 0 \\
0 & -6.79E+2 & 0 & 0 & 0 & -3.39E+4 & 0 & 6.79E+2 & 0 & 0 & 0 & -3.39E+4 \\
0 & 0 & -6.79E+2 & 0 & 3.39E+4 & 0 & 0 & 0 & 6.79E+2 & 0 & 3.39E+4 & 0 \\
0 & 0 & 0 & -3.79E+5 & 0 & 0 & 0 & 0 & 0 & 3.79E+5 & 0 & 0 \\
0 & 0 & -3.39E+4 & 0 & 1.11E+6 & 0 & 0 & 0 & 3.39E+4 & 0 & 2.28E+6 & 0 \\
0 & 3.39E+4 & 0 & 0 & 0 & 1.11E+6 & 0 & -3.39E+4 & 0 & 0 & 0 & 2.28E+6
\end{array}\right|
\end{array}$$

And when the offset of the previous NASTRAN code is taken into account, the stiffness matrix has the following values.

$$[K_{offset}]=\begin{array}{c}
\begin{array}{cccccccccccc}
u_1 & v_1 & w_1 & \theta_{x1} & \theta_{y1} & \theta_{z1} & u_2 & v_2 & w_2 & \theta_{x2} & \theta_{y2} & \theta_{z2} \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow
\end{array} \\
\left|\begin{array}{cccccccccccc}
7.00E+4 & 0 & 0 & 0 & 3.85E+5 & 0 & -7.00E+4 & 0 & 0 & 0 & -3.85E+5 & 0 \\
0 & 6.79E+2 & 0 & -3.73E+3 & 0 & 3.39E+4 & 0 & -6.79E+2 & 0 & 3.73E+3 & 0 & 3.39E+4 \\
0 & 0 & 6.79E+2 & 0 & -3.39E+4 & 0 & 0 & 0 & -6.79E+2 & 0 & -3.39E+4 & 0 \\
0 & -3.73E+3 & 0 & 4.00E+5 & 0 & -1.87E+5 & 0 & 3.73E+3 & 0 & -4.00E+5 & 0 & -1.87E+5 \\
3.85E+5 & 0 & -3.39E+4 & 0 & 4.40E+6 & 0 & -3.85E+5 & 0 & 3.39E+4 & 0 & -1.00E+6 & 0 \\
0 & 3.39E+4 & 0 & -1.87E+5 & 0 & 2.28E+6 & 0 & -3.39E+4 & 0 & 1.87E+5 & 0 & 1.11E+6 \\
-7.00E+4 & 0 & 0 & 0 & -3.85E+5 & 0 & 7.00E+4 & 0 & 0 & 0 & 3.85E+5 & 0 \\
0 & -6.79E+2 & 0 & 3.73E+3 & 0 & -3.39E+4 & 0 & 6.79E+2 & 0 & -3.73E+3 & 0 & -3.39E+4 \\
0 & 0 & -6.79E+2 & 0 & 3.39E+4 & 0 & 0 & 0 & 6.79E+2 & 0 & 3.39E+4 & 0 \\
0 & 3.73E+3 & 0 & -4.00E+5 & 0 & 1.87E+5 & 0 & -3.73E+3 & 0 & 4.00E+5 & 0 & 1.87E+5 \\
-3.85E+5 & 0 & -3.39E+4 & 0 & -1.00E+6 & 0 & 3.85E+5 & 0 & 3.39E+4 & 0 & 4.40E+6 & 0 \\
0 & 3.39E+4 & 0 & -1.87E+5 & 0 & 1.11E+6 & 0 & -3.39E+4 & 0 & 1.87E+5 & 0 & 2.28E+6
\end{array}\right|
\end{array}$$

As a result of the offset introduced, some new coefficients appear, and some change their value.

To study the structure, the beam is clamped in the node 1. Therefore, only the submatrix highlighted has to be considered.

The previous structures are loaded with a unitary rotation in the θ_{y2}. With that displacement, the element load in the same direction (bending moment) have the following values:

$$M_{y2} = 2.28E + 6$$
$$M_{y2\_offset} = 4.40E + 6$$

The forces values are different. This difference has been explained in the previous chapter, and it is the consequence of the "Steiner effect" introduced by the offset.

After obtaining these results, it is possible to think that if the structures are loaded by means of a bending moment (M_{y2}), the displacement will be different.

The best way to check it, it is computing the flexibility matrix and applying the bending moment.

$$
[F]= \begin{array}{cccccc}
u_2 & v_2 & w_2 & \theta_{x2} & \theta_{y2} & \theta_{z2} \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow
\end{array}
$$

$$
[F]=\left|\begin{array}{cccccc}
1.43E\text{-}5 & 0 & 0 & 0 & 0 & 0 \\
0 & 5.76E\text{-}3 & 0 & 0 & 0 & 8.57E\text{-}5 \\
0 & 0 & 5.76E\text{-}3 & 0 & -8.57E\text{-}5 & 0 \\
0 & 0 & 0 & 2.64E\text{-}6 & 0 & 0 \\
0 & 0 & -8.57E\text{-}5 & 0 & 1.71E\text{-}6 & 0 \\
0 & 8.57E\text{-}5 & 0 & 0 & 0 & 1.71E\text{-}6
\end{array}\right|
$$

$$
\begin{array}{cccccc}
u_2 & v_2 & w_2 & \theta_{x2} & \theta_{y2} & \theta_{z2} \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow
\end{array}
$$

$$
[F_{offset}]=\left|\begin{array}{cccccc}
6.61E\text{-}5 & 0 & 4.71E\text{-}4 & 0 & -9.43E\text{-}6 & 0 \\
0 & 5.84E\text{-}3 & 0 & 1.45E\text{-}5 & 0 & 8.57E\text{-}5 \\
4.71E\text{-}4 & 0 & 5.76E\text{-}3 & 0 & -8.57E\text{-}5 & 0 \\
0 & 1.45E\text{-}5 & 0 & 2.64E\text{-}6 & 0 & 0 \\
-9.43E\text{-}6 & 0 & -8.57E\text{-}5 & 0 & 1.71E\text{-}6 & 0 \\
0 & 8.57E\text{-}5 & 0 & 0 & 0 & 1.71E\text{-}6
\end{array}\right|
$$

If the structure is loaded by means of a unitary bending moment. The rotation θ_{y2} of the structure will have the same value for both structures.

$$\theta_{y2} = 1.71E-6$$
$$\theta_{y2\_offset} = 1.71E-6$$

Therefore, it is possible to deduce that the flexibility components are not necessary the inverse of the corresponding stiffness component.

Summarising the previous analysis:

Unitary rotation $\theta_{y2} = 1$
- → No-offset structure → Bending moment $M_{y2} = 2.28E+6$
- → Offset structure → Bending moment $M_{y2\_offset} = 4.40E+6$

Unitary bending moment $M_{y2} = 1$
- → No-offset structure → Rotation $\theta_{y2} = 1.71E+6$
- → Offset structure → Rotation $\theta_{y2\_offset} = 1.71E+6$

Thus, the previous values represent the stiffness and flexibility components of the matrixes. These components have been highlighted in the matrixes.

Comparing the values, it is possible to see that the flexibility bending components are the same with and without offset. But on the other hand, the stiffness components are different.

As a conclusion, it is really important to understand that the flexibility and stiffness components are different. Most of the time, to see the stiffness of a component, the structure is loaded with forces. But this is not a good practice since only the flexibility can be obtained in this way. So it can lead to confusion with the stiffness values.